Strategy-proof and fair assignment is wasteful

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I prove there exists no assignment mechanism that is strategy-proof, non-wasteful and satisfies equal treatment of equals. When outside options may exist, this strengthens the impossibility result of Bogomolnaia and Moulin (2001) by weakening ordinal efficiency to non-wastefulness. My result solves an open question posed by Erdil (2014) and complements his results on the efficient frontier of random assignment mechanisms.

Keywords: random assignment, random serial dictatorship, strategy-proofness, ordinal efficiency, non-wastefulness. *JEL*: C78, D47, D61.

1 Introduction

In the assignment problem a number of heterogeneous, indivisible objects are to be distributed among several agents, with each agent entitled to at most one object.[1] There are no priorities and randomization is used to ensure fairness. Monetary transfers are disallowed. A *mechanism* elicits ordinal preferences of agents and outputs a random assignment of objects to agents.

I prove that if a mechanism is *strategy-proof* (truthfulness is a dominant strategy) and *fair* (equal treatment of equals: agents who report the same preferences face the same lottery over objects), then it is necessarily *wasteful*. A mechanism is wasteful if there exists an object x that is unassigned with positive probability and an agent who prefers x to another object (or the outside option) that she receives with positive probability.

Non-wastefulness is an ex-ante efficiency concept that is weaker than the standard notion of *ordinal efficiency* (not being first-order stochastically dominated). Bogomolnaia and Moulin (2001) prove that all strategy-proof and fair mechanisms are ordinally inefficient. My result strengthens theirs in the general setting where outside options may exist.

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[1] An important real-world application is housing assignment (public housing, campus housing, etc.). The assignment problem is also a building block of more complex matching problems, such as school choice (many-to-one matching, and priorities may be present) and course allocation (many-to-many matching).

As an illustration, suppose there are four agents $i = 1,2,3,4$ who report their true strict preferences $\succ^i$ over objects a, b, c and the outside option $\emptyset$. The canonical Random Serial Dictatorship (RSD) procedure[2] induces the *random assignment* shown center-left, a matrix whose row i shows the lottery over (a,b,c) faced by agent i.[3] Two other random assignments P_2 and P_3 are also shown.

Preferences	RSD (wasteful)			P_2 (non-wasteful)			P_3 (ordinally efficient)		
$a \succ^1 b \succ^1 c \succ^1 \emptyset$	5/12	1/12	5/12	5/12	1/12	1/2	1/2	0	1/2
$a \succ^2 b \succ^2 c \succ^2 \emptyset$	5/12	1/12	5/12	5/12	1/12	1/2	1/2	0	1/2
$b \succ^3 a \succ^3 \emptyset \succ^3 c$	1/12	5/12	0	1/12	5/12	0	0	1/2	0
$b \succ^4 a \succ^4 \emptyset \succ^4 c$	1/12	5/12	0	1/12	5/12	0	0	1/2	0

In the random assignment induced by RSD, object c is wasted: with probability 1/6 it is unassigned, yet agents 1 and 2 prefer c to receiving the outside option, which occurs with probability 1/12 each. P_2, where agents 1 and 2 receive c with probability 1/2, is a non-wasteful improvement over RSD. Still, it is ordinally inefficient: there are other random assignments, for example P_3, that first-order stochastically dominate it according to the true preferences.

The paper is organized as follows. In Section 2, I present the model. In Section 3, I state and prove the impossibility theorem. In Section 4, I verify minimality of the theorem's assumptions and discuss the importance of outside options. In Section 5, I conclude by discussing the relationship between waste and the set of undominated strategy-proof mechanisms.

2 Model

Let $N = \{1,2,\ldots,n\}$ be a set of agents and $O = \{a,b,c,\ldots\}$ a set of m objects. Each agent $i \in N$ has strict preferences $\succ_i$ over O and the outside option $\emptyset$. Objects less preferred than the outside option are said to be *unacceptable*. Preferences $a \succ_i b \succ_i \emptyset \succ_i c$ (for example) will be represented compactly as a list $R^i = ab$; unacceptable objects are omitted from the list, as their ordering is irrelevant. $R = (R^i)_{i \in N}$ is the profile of preferences for all agents in N. Let $\mathcal{R}$ denote the set of all such possible profiles.

A *(random) assignment* is a matrix $P = (P_{ix})_{i \in N, x \in O}$, with rows indexed by agents $i \in N$ and columns indexed by objects $x \in O$. For each i and x, $P_{ix} \in [0,1]$ is the probability that agent i receives object x. *Agent feasibility* holds if $\sum_{x \in O} P_{ix} \leq 1$ for each $i \in N$. *Object feasibility* holds if $\sum_{i \in N} P_{ix} \leq 1$ for each $x \in O$. P is *individually rational* with respect to preference profile $R = (R^i)_{i \in N}$ if $\emptyset \succ_i x$ implies $P_{ix} = 0$, for all $i \in N$ and $x \in O$.

[2] Randomly choose an ordering over agents; in this order, assign agents to their preferred object among those that have not yet been assigned. RSD is strategy-proof, fair, and ex-post efficient (Abdulkadiroğlu and Sönmez, 1998).

[3] This approach was first employed by Hylland and Zeckhauser (1979), who show that it is always possible to decompose a random assignment as a lottery over deterministic assignments. Random assignment matrices are particularly well-suited to the strategic and ex-ante welfare analyses carried out in this paper.

A random assignment is *feasible* if it satisfies agent feasibility, object feasibility, and individual rationality. Let Π denote the set of all feasible random assignments. A generalization of the Birkhoff–von Neumann theorem (e.g. Kojima and Manea, 2010) ensures that all feasible random assignments can be decomposed as lotteries over deterministic assignments.

With respect to preference profile $R = (R^i)_{i \in N}$, a random assignment P:

- is *fair* or satisfies *equal treatment of equals* if $R^i = R^j$ implies $P_{ix} = P_{jx}$ for all $x \in O$;
- is *wasteful* if there exist $i \in N$, $x \in O$, and $y \in O \cup \{\emptyset\}$ such that $x \succ_i y$, $\sum_{i \in N} P_{ix} < 1$ and $P_{iy} > 0$. In words, x is wasted if it is unassigned with positive probability and there is an agent i who prefers it to an object (or the outside option) y that she receives with positive probability. P is *non-wasteful* if it is not wasteful;
- is *ordinally inefficient* if there exists another assignment $P' \neq P$ such that for all $i \in N$, the lottery over objects $(P'_{ix})_{x \in O}$ first-order stochastically dominates $(P_{ix})_{x \in O}$ according to R. In this case, we say that P' *dominates* P. If there is no such P', then P is *ordinally efficient*.

Ordinal efficiency implies non-wastefulness. To see this, let assignment P be wasteful, say at (i, x, y) as above. Then it is ordinally inefficient, because it is dominated by the assignment P' that takes P and moves probability mass $\min\{P_{iy}, 1 - \sum_{i \in N} P_{ix}\}$ from P_{iy} to P_{ix}.

An *(assignment) mechanism* is a function $P : \mathcal{R} \to \Pi$ that maps preference profiles $R \in \mathcal{R}$ into random assignments $P(R) \in \Pi$. A mechanism is individually rational; satisfies equal treatment of equals; is non-wasteful; is ordinally efficient, if for every $R \in \mathcal{R}$, $P(R)$ has that property.

A mechanism P is *strategy-proof* if for every agent i with preferences $\succcurlyeq^i$ represented by preference list R^i, every profile $R = (R^i, R^{-i})$, and every profile $R' = (\hat{R}^i, R^{-i})$ where i deviates to $\hat{R}^i$, the allocation (lottery over objects) that i receives at R first-order stochastically dominates the allocation at R' according to the true preferences $\succcurlyeq^i$. That is, for every object y the probability of receiving an object that is at least as good as y is at least as large when reporting the true preference list R^i than when reporting any other preference list $\hat{R}^i$:

$$\text{for every } y \in O, \quad \sum_{x \,:\, x \succcurlyeq^i y} P_{ix}(R) \geq \sum_{x \,:\, x \succcurlyeq^i y} P_{ix}(R').$$

3 Impossibility theorem

My main contribution is the following impossibility theorem, which solves an open question posed by Erdil (2014).

Theorem. *Let there be $n \geq 4$ agents and $m \geq 3$ objects. Then there is no mechanism that is strategy-proof, non-wasteful and satisfies equal treatment of equals.*

The proof is given in section 3.2. Before that, section 3.1 collects five facts that will be used repeatedly in the main proof.

3.1 Five facts

Let $N = \{1,2,3,4\}$ and $O = \{a,b,c\}$. Let P be a strategy-proof (SP) mechanism, and let $R = (R^i, R^{-i})$ and $R' = (R'^i, R^{-i})$ be preference profiles that differ only for agent i.

Fact 1. *If R^i and R'^i list the same object x first, then $P_{ix}(R) = P_{ix}(R')$.*

Fact 2. *If R^i and R'^i list the same two objects x and y as highest (not necessarily in the same order), then $P_{ix}(R) + P_{iy}(R) = P_{ix}(R') + P_{iy}(R')$.*

Fact 3. *If R^i and R'^i list all objects a, b, c as acceptable (in any order), then $P_{ia}(R) + P_{ib}(R) + P_{ic}(R) = P_{ia}(R') + P_{ib}(R') + P_{ic}(R')$.*

Proof of Facts 1, 2, and 3. To prove Fact 1, fix R^{-i}, and suppose R^i is i's true preference list. By deviating to R'^i, i would obtain $P_{ix}(R')$ of x. By SP, the truthful allocation of her preferred object must be no less than this: $P_{ix}(R) \geq P_{ix}(R')$.

Vice versa, suppose R'^i is i's true preference list. The symmetric logic applies—deviating to R^i cannot give i more of her preferred object (which is still x), hence $P_{ix}(R') \geq P_{ix}(R)$. The two inequalities combined give $P_{ix}(R') = P_{ix}(R)$. Facts 2 and 3 follow from the same argument, replacing $P_{ix}(\cdot)$ with the sum over the two most preferred and all objects respectively. □

Now suppose that P also satisfies non-wastefulness (NW) and equal treatment of equals (ETE), in addition to SP.

Fact 4. *Let $R = (R^1, R^2, R^3, R^4)$ be a preference profile where all 4 agents list the same object, say a, as highest. Then $P_{ia}(R) = 1/4$ for all agents i.*

Proof. First consider the profile $R_0 = (a, a, a, a)$. By NW and ETE, $P_{ia}(R_0) = 1/4$ for all i. Now consider $R_1 = (R^1, a, a, a)$. By Fact 1, agent 1 must get the same allocation of her preferred object a at the two profiles R_0 and R_1, hence $P_{1a}(R_1) = 1/4$. Object a cannot be wasted (it is the most preferred object by at least one agent); hence, by NW and ETE, $P_{2a}(R_1) = P_{3a}(R_1) = P_{4a}(R_1) = 1/4$. Now consider $R_2 = (a, R^2, a, a)$. As before, Fact 1 gives $P_{2a}(R_2) = 1/4$ and NW and ETE give $P_{1a}(R_2) = P_{3a}(R_2) = P_{4a}(R_2) = 1/4$. The same logic shows that $R_3 = (a, a, R^3, a)$ and $R_4 = (a, a, a, R^4)$ are uniformly $1/4$.

Now consider $R_{12} = (R^1, R^2, a, a)$. Fact 1 applied to agent 1 and profile R_2 gives $P_{1a}(R_{12}) = P_{1a}(R_2) = 1/4$. Symmetrically, Fact 1 applied to agent 2 and profile R_1 gives $P_{2a}(R_{12}) = P_{2a}(R_1) = 1/4$. Next, NW and ETE give $P_{3a}(R_{12}) = P_{4a}(R_{12}) = 1/4$. The same logic shows that all profiles R_{ij} defined similarly are uniformly $1/4$.

Now consider $R_{123} = (R^1, R^2, R^3, a)$. Fact 1 applied to agent 1 and profile R_{23} gives $P_{1a}(R_{123}) = P_{1a}(R_{23}) = 1/4$. In the same fashion we obtain $P_{2a}(R_{123}) = P_{1a}(R_{13}) = 1/4$ and $P_{2a}(R_{123}) = P_{1a}(R_{13}) = 1/4$. NW then gives $P_{4a}(R_{123}) = 1/4$. The same argument shows that allocations at R_{124}, R_{134}, and R_{234} are uniformly $1/4$.

Finally, consider $R_{1234} = R$. Fact 1 applied to agent 1 and profile R_{234} gives $P_{1a}(R) = P_{1a}(R_{234}) = 1/4$, and symmetrically for other agents. Thus $P_{ia}(R) = 1/4$ for all i. □

Fact 5. *Let $R = (R^1, R^2, R^3, R^4)$ be a preference profile where all 4 agents either list $a \in O$ as first choice or as unacceptable. Suppose $k \geq 1$ agents list object a first; then they receive $P_{ia}(R) = 1/k$.*

Proof. Retrace the argument of the previous proof, this time starting from the profile R_0 that replaces preferences of agents who deem x acceptable with the list "a", and leaves other lists unchanged. At R_0, by NW and ETE each of the k agents receives $1/k$ of a, as all other agents must receive 0 of a by individual rationality. At each further step until R is reached, consider all profiles obtained by replacing one "a" with the true preferences; at each of these intermediate profiles, all k agents receive $1/k$ of a. □

3.2 Proof of impossibility theorem

The outline of the proof is as follows. First, assume that a strategy-proof, fair and non-wasteful mechanism exists. Second, note that the restriction to a corner of the preference domain also satisfies these properties. Then, using the facts of section 3.1 and direct consequences of the assumptions, progressively pin down the mechanism at different preferences profiles. This sequence of implications eventually leads to a contradiction.

Proof of Theorem. Suppose there exists a mechanism P that is feasible (that is, satisfies agent feasibility (AF), object feasibility (OF), and individual rationality (IR)) and satisfies strategy-proofness (SP), equal treatment of equals (ETE), and non-wastefulness (NW).

Consider the following subset of the full preference domain: agents $i > 4$ consider all objects unacceptable, and agents $i \in \{1, 2, 3, 4\}$ consider objects $x \notin \{a, b, c\}$ unacceptable. Then IR reduces the assignment problem to the first 4 agents and first 3 objects. Since P satisfies SP, NW and ETE on the full preference domain, it also does so on this restricted preference domain. Thus, it is enough to consider the case $n = 4$, $m = 3$.

Let $N = \{1, 2, 3, 4\}$ and $O = \{a, b, c\}$. Consider the random assignment $P(R_0)$ at preference profile $R_0 = (abc, abc, a, cab)$, depicted below in matrix form. IR immediately gives $P_{3b}(R_0) = P_{3c}(R_0) = 0$. Steps A through E will progressively fill in $P(R_0)$ using information at auxiliary preference profiles. Step F will then use $P(R_0)$ to derive a contradiction with the assignment at profile $R_1 = (abc, abc, a, acb)$.

R_0	a	b	c
abc	$1/3$ **A**	.	.
abc	$1/3$ **A′**	.	.
a	$1/4$ **B,D**	0	0
cab	$1/12$ **D**	$1/6$ **E**	$2/3$ **C**

Random assignment $P(R_0)$. Each row represents one agent. The first column shows that agent's preference list; the other three show the probability of receiving a, b, c respectively.

Superscript capital letters indicate the step of the proof at which the cell is filled in.

Step A. Consider profiles $A = (a, abc, a, ca)$, $A' = (a, a, a, cab)$, and $A'' = (a, abc, a, cab)$. The assignments at these profiles are depicted below. IR implies that certain cells must be zero. Next, in steps *i* through *vii*, the assumptions and five facts of Section 3.1 are used to fill in certain cells.

A	a	b	c
a	·	0	0
abc	·	·	0^{i}
a	·	0	0
ca	·	0	1^{ii}

A'	a	b	c
a	$1/3^{iv}$	0	0
a	$1/3^{iv}$	0	0
a	$1/3^{iv}$	0	0
cab	0^{iii}	0^{iii}	1^{iii}

A''	a	b	c
a	$1/3^{vii}$	0	0
abc	$1/3^{vi}$	·	·
a	$1/3^{vii}$	0	0
cab	0^{v}	0^{v}	1^{v}

Random assignments $P(A)$, $P(A')$, and $P(A'')$. Superscript Roman numerals refer to the sub-step at which the cell is filled in. Cells denoted with · are not pinned down or are not relevant.

i Suppose that $P_{2c}(A) > 0$. By AF, $P_{2b}(A) < 1$. Since b is acceptable only to agent 2, $\sum_i P_{ib}(A) = P_{2b}(A) < 1$. Thus b is not fully allocated and agent 2 prefers it to c, which she is allocated with positive probability. This violates NW of b. Therefore $P_{2c}(A) = 0$.

ii No agent other than 4 is allocated c, which is her preferred object. By NW, $P_{4c}(A) = 1$.

iii Now consider A'. By NW of c, $P_{4c}(A') = 1$ and thus $P_{4a}(A') = P_{4b}(A') = 0$.

iv NW and ETE then imply $P_{1a}(A') = P_{2a}(A') = P_{3a}(A') = 1/3$.

v At profiles A and A'', all preferences expect those of agent 4 are the same, and 4 ranks c first at both profiles. By Fact 1, $P_{4c}(A'') = P_{4c}(A) = 1$, and thus $P_{4a}(A'') = P_{4b}(A'') = 0$.

vi A' and A'' are equal except for agent 2, who ranks a first at both profiles. By Fact 1, $P_{2a}(A'') = P_{2a}(A') = 1/3$.

vii At A'' object a cannot be wasted, as it is the preferred object of an agent. Thus, agents 1 and 3 must share the remaining probability $1 - P_{2a}(A'') = 2/3$, and by ETE they do so equally: $P_{1a}(A'') = P_{3a}(A'') = 1/3$.

To conclude Step A, note that R_0 and A'' are equal except for agent 1, who ranks a first at both profiles. By Fact 1, $P_{1a}(R_0) = P_{1a}(A'') = 1/3$.

Step A′. At R_0, agents 1 and 2 report the same preferences. By ETE, $P_{2a}(R_0) = P_{1a}(R_0) = 1/3$.

Step B. Consider profiles $B = (abc, abc, abc, abc)$ and $B' = (abc, abc, abc, cab)$. Step B will show that $P_{3a}(R_0) \geq 1/4$.

B	a	b	c
abc	$1/4^{i}$	1/4	1/4
abc	1/4	1/4	1/4
abc	1/4	1/4	1/4
abc	1/4	1/4	1/4

B'	a	b	c
abc	$\geq 1/4^{iii}$	·	·
abc	$\geq 1/4^{iii}$	·	·
abc	$\geq 1/4^{iii}$	·	·
cab	$\leq 1/4^{ii}$	·	·

R_0	a	b	c
abc	1/3	·	·
abc	1/3	·	·
a	$\geq 1/4^{\mathbf{B}}$	0	0
cab	·	·	·

i At profile B, by ETE all agents must receive the same allocation. For each object, any allocation less than 1/4 per agent would be wasteful, and more than 1/4 would violate OF. Therefore $P(B)$ is uniformly 1/4.

ii Again at B, by SP, agent 4's assignment when deviating to $\hat{R}^4 = cab$ must be FOSD by her assignment at the true profile B. In particular, at $B' = (B^{-4}, \hat{R}^4)$, she must receive no more of a (her first choice under true preferences abc) than at B: $P_{4a}(B') \leq P_{4a}(B) = 1/4$.

iii At B', a cannot be wasted. Since the remaining three profiles are equal, by ETE each must receive no less than $\frac{1}{3}(1 - 1/4) = 1/4$ of a.

To conclude Step B, note that R_0 and B' are equal except for agent 3, who ranks a first at both profiles. By Fact 1, $P_{3a}(R_0) = P_{3a}(B') \geq 1/4$.

Step C. Consider profiles $C = (a, abc, a, c)$, $C' = (abc, c, a, c)$, $C'' = (abc, cba, a, c)$, and $C''' = (abc, abc, a, c)$. Step C will show that $P_{4c}(R_0) = 2/3$.

C	a	b	c
a	$1/2^{iii}$	0	0
cba	0^{ii}	$1/2^{ii}$	$1/2^{i}$
a	$1/2^{iii}$	0	0
c	0	0	$1/2^{i}$

C'	a	b	c
abc	$1/2^{i}$	$1/2^{ii}$	0^{ii}
c	0	0	$1/2^{iii}$
a	$1/2^{i}$	0	0
c	0	0	$1/2^{iii}$

i At C, since for agents 2 and 4 c is the best object and c is unacceptable for other agents, Fact 5 implies $P_{2c}(C) = P_{4c}(C) = 1/2$.

Symmetrically, at C' for agents 1 and 3 a is the best object and a is unacceptable for other agents, hence Fact 5 implies $P_{1a}(C') = P_{3a}(C') = 1/2$.

ii At C, since b is agent 2's next most preferred object and b is unacceptable for other agents, NW implies $P_{2b}(C) = 1/2$ and hence $P_{2a}(C) = 0$.

Symmetrically, at C' b is agent 1's next preferred object and b is unacceptable for other agents, so NW implies $P_{1b}(C') = 1/2$ and hence $P_{1c}(C') = 0$.

iii At C, NW of a and ETE together imply that $P_{1a}(C) = P_{3a}(C) = 1/2$.

Symmetrically, at C', NW of c and ETE imply that $P_{2c}(C') = P_{4c}(C') = 1/2$.

C''	a	b	c
abc	$1/2^{iv}$	$1/2^{vi}$	0^{vi}
cba	0^{vi}	$1/2^{vi}$	$1/2^{v}$
a	$1/2^{vii}$	0	0
c	0	0	$1/2^{vii}$

C'''	a	b	c
abc	$1/3^{viii}$	$1/2^{ix}$	$1/6^{xi}$
abc	$1/3^{viii}$	$1/2^{ix}$	$1/6^{x}$
a	$1/3^{vii}$	0	0
c	0	0	$2/3^{xii}$

R_0	a	b	c
abc	$1/3$	·	·
abc	$1/3$	·	·
a	$\geq 1/4$	0	0
cab	·	·	$2/3^{\mathbf{C}}$

iv At C'', agent 1 has the same most preferred object a as at C, and all other preferences are the same. By Fact 1, $P_{1a}(C'') = P_{1a}(C) = 1/2$.

v At C'', agent 2 has the same most preferred object c as at C', and all other preferences are the same. By Fact 1, $P_{2c}(C'') = P_{2c}(C') = 1/2$.

vi By AF, $P_{1b}(C'') \leq 1/2$ and $P_{2b}(C'') \leq 1/2$. b is the next most preferred object for both 1 and 2, and since there is enough of b to satisfy both agents, NW implies $P_{1b}(C'') = 1/2$ and $P_{2b}(C'') = 1/2$. Then AF implies $P_{1c}(C'') = 0$ and $P_{2a}(C'') = 0$.

vii By NW objects a and c must be fully allocated, hence $P_{3a}(C'') = 1/2$ and $P_{4c}(C'') = 1/2$.

viii Now consider C'''. By Fact 5, $P_{1a}(C''') = P_{2a}(C''') = P_{3a}(C''') = 1/3$.

ix By NW and ETE, $P_{1b}(C''') = P_{2b}(C''') = 1/2$.

x C'' and C''' are equal except for agent 2, who lists all objects as acceptable at both profiles. By Fact 3, $P_{2c}(C''') = \sum_{x \in O} P_{2x}(C'') - P_{2a}(C''') - P_{2b}(C''') = 1/6$.

xi By ETE, $P_{1c}(C''') = P_{2c}(C''') = 1/6$.

xii Finally, by NW the total assignment of c at C''' must be 1, hence $P_{4c}(C''') = 1 - P_{1c}(C''') - P_{2c}(C''') - P_{3c}(C''') = 2/3$.

To conclude Step C, note that at profiles R_0 and C''', c is the most preferred object for agent 4, and all other preferences are the same. By Fact 1, $P_{4c}(R_0) = P_{4c}(C''') = 2/3$.

Step D. Consider profile $D = (abc, abc, a, ac)$. Step D will show $P_{3a}(R_0) = 1/4$ and $P_{4a}(R_0) = 1/12$.

D	a	b	c
abc	$1/4^{i}$	$1/2^{ii}$	$\leq 1/4^{iii}$
abc	$1/4^{i}$	$1/2^{ii}$	$\leq 1/4^{iii}$
a	$1/4^{i}$	0	0
ac	$1/4^{i}$	0	$\geq 1/2^{iv}$

R_0	a	b	c
abc	$1/3$	·	·
abc	$1/3$	·	·
a	$1/4^{\mathbf{D}}$	0	0
cab	$1/12^{\mathbf{D}}$	·	$2/3$

i By Fact 4, $P_{ia}(D) = 1/4$ for all agents i.

ii By NW and ETE, $P_{1a}(D) = P_{2a}(D) = 1/2$.

iii By AF, $P_{1c} \leq 1/4$ and $P_{2c} \leq 1/4$.

iv Object c cannot be wasted: if its total allocation were strictly less than 1, than at least one out of agents 1, 2, and 4 would not have a full allocation, and they all prefer c over the outside option. Therefore $P_{4c} = 1 - P_{1c} - P_{2c} \geq 1 - 1/4 - 1/4 = \frac{1}{2}$.

At R_0 and D agent 4 lists a and c as her two preferred objects, and all other preferences are the same. By Fact 2, $P_{4a}(R_0) + P_{4c}(R_0) = P_{4a}(D) + P_{4c}(D) \geq 1/4 + 1/2 = 3/4$. Therefore $P_{4a}(R_0) \geq 3/4 - P_{4c}(R_0) = 3/4 - 2/3 = 1/12$.

Furthermore, by OF, at R_0 the sum over column a cannot be greater than 1. From Step B we know that $P_{3a}(R_0) \geq 1/4$. As $1/3 + 1/3 + 1/4 + 1/12 = 1$, this implies $P_{3a}(R_0) = 1/4$ and $P_{4a}(R_0) = 1/12$.

Step E. Consider profile $E = (abc, abc, a, abc)$. Step E will show that $P_{4b}(R_0) = 1/6$.

E	a	b	c
abc	$1/4^{i}$	$1/3^{ii}$	$1/3^{ii}$
abc	$1/4^{i}$	$1/3^{ii}$	$1/3^{ii}$
a	$1/4^{i}$	0	0
abc	$1/4^{i}$	$1/3^{ii}$	$1/3^{ii}$

R_0	a	b	c
abc	1/3	·	·
abc	1/3	·	·
a	1/4	0	0
cab	1/12	$1/6^{\mathbf{E}}$	2/3

i By Fact 4, $P_{ia}(E) = 1/4$ for all agents i.

ii By NW and ETE, $P_{1b}(E) = P_{2b}(E) = P_{4b}(E) = 1/3$, $P_{1c}(E) = P_{2c}(E) = P_{4c}(E) = 1/3$.

By Fact 3, agent 4 must receive the same total allocation at E and R_0. Therefore

$$P_{4b}(R_0) = P_{4a}(E) + P_{4b}(E) + P_{4c}(E) - P_{4a}(R_0) - P_{4c}(R_0) = \frac{3+4+4-1-8}{12} = \frac{1}{6}.$$

Step F. Consider profiles $F = (ab, ab, a, acb)$, $F' = (ab, abc, a, acb)$, and $R_1 = (abc, abc, a, acb)$.

F	a	b	c
ab	$1/4^{i}$	$1/2^{iii}$	0
ab	$1/4^{i}$	$1/2^{iii}$	0
a	$1/4^{i}$	0	0
acb	$1/4^{i}$	0^{ii}	$3/4^{ii}$

F'	a	b	c
ab	$1/4^{i}$	$1/2^{vi}$	0
abc	$1/4^{i}$	$1/2^{iv}$	$1/4^{v}$
a	$1/4^{i}$	0	0
acb	$1/4^{i}$	0^{v}	$3/4^{v}$

i At all three profiles F, F', and R_1, Fact 4 implies the allocation of a is 1/4 for all agents.

ii At F, only agent 4 demands c, and it is her next most preferred object. NW then implies that $P_{4c}(F) = 3/4$, and hence $P_{4b}(F) = 0$.

iii Next, NW and ETE imply $P_{1b}(F) = P_{2b}(F) = 1/2$.

iv At profiles F and F' all preferences other than agent 2's are equal. Agent 2 receives the same allocation of her preferred object a, and her second most preferred object b is the same at the two profiles. Thus, Fact 2 implies $P_{2b}(F') = P_{2b}(F) = 1/2$.

v By AF for agent 2, $P_{2c}(F') \leq 1/4$. By AF for agent 4, $P_{4c}(F') \leq 3/4$. Both of these constraints can be satisfied with equality, and anything less would violate NW. Therefore $P_{2c}(F') = 1/4$ and $P_{4c}(F') = 3/4$, which implies $P_{4b}(F') = 0$.

vi By NW, $P_{1b}(F') = 1/2$.

R_1	a	b	c
abc	$1/4$ ^{i}	$1/2$ ^{vii}	.
abc	$1/4$ ^{i}	$1/2$ ^{viii}	.
a	$1/4$ ^{i}	0	0
acb	$1/4$ ^{i}	0^{viii}	$\rightarrow\leftarrow$

R_0	a	b	c
abc	$1/3$	.	.
abc	$1/3$	.	.
a	$1/4$	0	0
cab	$1/12$	$1/6$	$2/3$

vii At profiles R_1 and F' all preferences other than agent 1's are equal. Agent 1 receives the same allocation of her preferred object a, and her second most preferred object b is the same at the two profiles. Thus, Fact 2 implies $P_{1b}(R_1) = P_{1b}(F') = 1/2$.

viii By ETE, $P_{2b}(R_1) = P_{1b}(R_1) = 1/2$. By OF, $P_{4b}(R_1) = 0$.

We are ready to obtain the desired contradiction at agent 4's allocation of c at R_1, $P_{4c}(R_1)$. Profile R_0 is the same as R_1 for agents other than 4. Furthermore, at R_0 and R_1 agent 4 agrees that a and c are the best two objects (in some order). By Fact 2,

$$P_{4a}(R_1) + P_{4c}(R_1) = P_{4a}(R_0) + P_{4c}(R_0) = \frac{1}{12} + \frac{2}{3} = \frac{3}{4}.$$

Therefore $P_{4c}(R_1) = 3/4 - P_{4a}(R_1) = 1/2$.

At the same time, since at R_0 and R_1 agent 4 considers all objects acceptable, by Fact 3 the total allocation must be the same:

$$P_{4a}(R_1) + P_{4b}(R_1) + P_{4c}(R_1) = P_{4a}(R_0) + P_{4a}(R_0) + P_{4c}(R_0) = \frac{1}{12} + \frac{1}{6} + \frac{2}{3} = \frac{11}{12}.$$

This implies $P_{4c}(R_1) = 11/12 - P_{4a}(R_1) - P_{4b}(R_1) = 2/3 \neq 1/2$. Contradiction. □

4 Complements and discussion

4.1 Logical independence

Any two of the theorem's three axioms can be simultaneously satisfied:

- serial dictatorship (for a fixed ordering of agents) is strategy-proof and non-wasteful (non-wastefulness reduces to ex-post efficiency in deterministic mechanisms), but does not satisfy equal treatment of equals;
- not assigning *any* object is strategy-proof and satisfies equal treatment of equals, but is clearly wasteful;
- the Probabilistic Serial mechanism of Bogomolnaia and Moulin (2001) satisfies equal treatment of equals and is non-wasteful (indeed, it is ordinally efficient), but is not strategy-proof.

4.2 Possibility results for small markets

I show that if there are $m \leq 2$ objects or $n \leq 3$ agents, there exists a mechanism which is strategy-proof, non-wasteful, and satisfies equal treatment of equals: namely, random serial dictatorship. As the impossibility theorem holds when $m \geq 3$ and $n \geq 4$, this gives a full characterization.

For any m and n, RSD is strategy-proof (it is an exogenous randomization over serial dictatorships, which are clearly strategy-proof) and satisfies equal treatment of equals (since the randomization over orderings is uniform). Furthermore:

Proposition 1. *When $m \leq 2$, RSD is non-wasteful.*

Proof. Without loss of generality, assume that all objects are acceptable to at least one agent. If $O = \{a\}$ (case $m = 1$), the RSD procedure always allocates a, hence it is not wasted.

Now let $O = \{a, b\}$ (case $m = 2$). If each object is acceptable to at least 2 agents, RSD always allocates both objects, hence they are not wasted. Otherwise, suppose a is acceptable only to agent 1. Then 1 receives an object with probability one (a is available regardless of ordering), and receives a positive probability of b only if she prefers it to a. Therefore a is not wasted. The symmetric argument shows that b is not wasted either. □

Proposition 2. *When $n \leq 3$, RSD is non-wasteful.*

Proof. Consider an agent i with preference list R^i that begins with xyz (where y or both y and z can be the outside option). Under the RSD procedure, at most two objects are unavailable at i's turn, hence i never receives an item worse than z. Thus z and worse objects are not wasted for agent i. Furthermore, x is not wasted because it is always assigned (to i or to another agent).

For y to be wasted for agent i, it cannot be the top choice of any other agent (otherwise it would always be assigned) and i must receive z with positive probability (we just showed that i cannot receive worse). This can only happen if for some permutation of the (necessarily three) agents, the first agent receives x, the second agent j expresses preferences that begin with xy, and the third agent is i. Then there are two agents, i and j, whose preferences both begin with xy. This immediately implies that y is always assigned hence not wasted.

Thus, no object is wasted for agent i. Repeat the argument for all agents to conclude. □

The argument breaks down for $n = 4$ because it is then possible for an agent to receive their third choice when no other agent shares the same top two choices. For example, consider the profile $(R^1, R^2, R^3, R^4) = (xy, zy, z, x)$. In order 1234, y is unassigned. In order 4321, agent 1 is unassigned (their third choice). Hence, y is wasted.

4.3 Importance of outside options

The assumption that objects may be unacceptable (that is, agents may have outside options) is essential for my impossibility result. On the restricted preference domain where all preference lists are complete and there are as many objects as agents, the mechanism that assigns objects uniformly at random trivially satisfies strategy-proofness, fairness and non-wastefulness.

The impossibility result of Bogomolnaia and Moulin (2001) is stated with these restrictions. It naturally extends to the more general environment, where my result nests theirs. Kesten and Kurino (2015) discuss the importance of outside options in a related setting.

5 Conclusion

I have proven that strategy-proofness and equal treatment of equals are incompatible with non-wastefulness, a mild efficiency notion.

This begs the question as to what (weaker) notion of efficiency is appropriate. Ideally, such a notion would characterize one or more mechanisms on the (strategy-proof) *efficient frontier*: the set of strategy-proof and fair mechanisms that are not dominated by a strategy-proof mechanism. While a closure argument easily shows that the efficient frontier is non-empty, there is no known example of a mechanism that belongs to it.

Erdil (2014) highlights the important connection between waste and the efficient frontier. He shows that a strategy-proof mechanism can dominate another only by being less wasteful; improvements that just reshuffle probabilities cannot be strategy-proof. A corollary of this result is that any non-wasteful mechanism is automatically on the efficient frontier. My impossibility result shuts down this approach to finding an undominated mechanism.

Conversely, Erdil shows that being on the efficient frontier does not imply non-wastefulness by proving that all mechanisms that dominate random serial dictatorship (which he shows not to be on the efficient frontier) are wasteful. My result clarifies that this is not due to RSD being a particular starting point: *all* mechanisms on the efficient frontier are wasteful.

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